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CONSTRUCTION OF DEGENERATE q-CHANGHEE POLYNOMIALS WITH WEIGHT α AND ITS APPLICATIONS

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Abstract: The aim of the present paper is to deal with introducing a new family of Changhee polynomials which is called degenerate q-Changhee polynomials with weight α by using p-adic q-integral on $\mathbb{Z}_p$. From this definition, we obtain some new summation formulae and properties. We also introduce the degenerate q-Changhee polynomials of higher-order with weight α and obtain some new interesting results.

Keywords and Phrases: Degenerate q-Changhee polynomials and numbers with weight α; higher-order degenerate q-Changhee polynomials and numbers with weight α, Stirling numbers of the first and second kind.

2020 Mathematics Subject Classification: 11B68, 11B83, 11S80.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the filed of p-adic rational numbers and the completion of an algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\ .\ |_p$ is normalized by $|\ p\ |_p = \frac{1}{p}$. For $q, x \in \mathbb{C}_p$ with $|\ q-1\ |_p < p^{-\frac{1}{p-1}}$. We define the q-analogue of a number x to be $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let $C(\mathbb{Z}_p)$ be the space of continuous function on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, Kim introduced the fermionic p-adic q-integral $I_{-q}(f)$ on $\mathbb{Z}_p$ (see [11, 12, 13])

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{(see [8, 19])}. \tag{1.1}$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x$$

$$= \lim_{N\to\infty} \frac{1}{[dp^N]_{-q}} \sum_{a=0}^{d-1} \sum_{x=0}^{p^N-1} f(x+dx)(-q)^{a+dx},$$

where $d \in \mathbb{N}$ with $d \equiv 1(mod 2)$.

The q-Euler polynomials $E_{n,q}(x)$ are defined by the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows (see [5, 12, 13])

$$\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{1.2}$$

At the point $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called the q-Euler numbers.

From (1.2), we have

$$\int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q} = E_{n,q}(x), (n \geq 0). \tag{1.3}$$

Carlitz's type q-Changhee polynomials are defined by the generating function as follows (see [4])

$$\int_{\mathbb{Z}_p} (1+t)^{[x+y]_q} d\mu_{-q}(y) = \sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!}. \tag{1.4}$$

Letting $x = 0$, $Ch_n = Ch_{n,q}(0), (n \geq 0)$ are called the q-Changhee numbers. From (1.3) and (1.4), we have

$$\sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!} = \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} [x+y]_q^k d\mu_{-q}(y) \frac{1}{k!} (\log(1+t))^k$$

$$= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} [x+y]_q^k d\mu_{-q}(y) \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} E_{k,q}(x) S_1(n,k) \right) \frac{t^n}{n!}. \tag{1.5}$$

Thus, we have

$$Ch_{n,q}(x) = \sum_{m=0}^{n} E_{k,q}(x) S_1(n,k). \tag{1.6}$$

In [31], Ryoo introduced the q-Euler polynomials with weight α which can be represented by the p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\sum_{n=0}^{\infty} E_{n,q}^{(\alpha)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{t[x+y]_{q^\alpha}} d\mu_{-q}(y). \tag{1.7}$$

It is clear that, we have

$$E_{n,q}^{(\alpha)}(x) = \sum_{m=0}^{n} \binom{n}{m} q^{\alpha m x} [x]_{q^\alpha}^{n-m} E_{n,q}^{(\alpha)}. \tag{1.8}$$

The degenerate q-Euler polynomials are defined by means of the following generating function (see [5])

$$\sum_{n=0}^{\infty} E_{n,q,\lambda}^{(\alpha)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y). \tag{1.9}$$

When $x = 0$, $E_{n,q,\lambda}^{(\alpha)} = E_{n,q,\lambda}^{(\alpha)}(0)$ are called the degenerate q-Euler numbers.

The modified degenerate q Changhee polynomials are defined by means of the following generating function (see [6])

$$\int_{\mathbb{Z}_p} (1+\lambda \log(1+t))^{\frac{[x+y]_q}{\lambda}} d\mu_{-q}(y) = \sum_{n=0}^{\infty} Ch_{n,q,\lambda}(x) \frac{t^n}{n!}. \tag{1.10}$$

In the case $x = 0$, $Ch_{n,q,\lambda} = Ch_{n,q,\lambda}(0)$ are called the modified degenerate q Changhee numbers.

For $j \geq 0$, the Stirling numbers of the first kind are defined by

$$(\xi)_j = \sum_{l=0}^{j} S_1(j,l) \xi^l, \text{(see [1-10])}, \tag{1.11}$$

where $(\xi)_0 = 1$, and $(\xi)_j = \xi(\xi-1)\cdots(\xi-j+1), (j \geq 1)$. From (1.11), it is easy to see that

$$\frac{1}{r!}(\log(1+z))^r = \sum_{j=r}^{\infty} S_1(j,r) \frac{z^j}{j!}, \quad (r \geq 0), \text{(see [11-20])}. \tag{1.12}$$

For $j \geq 0$, the Stirling numbers of the second kind are defined by

$$\xi^j = \sum_{l=0}^{j} S_2(j,l)(\xi)_l, \text{(see [21-31])}. \tag{1.13}$$

From (1.13), we see that

$$\frac{1}{r!}(e^z - 1)^r = \sum_{j=r}^{\infty} S_2(j,r)\frac{z^j}{j!}. \tag{1.14}$$

In this paper, we define degenerate q-Changhee polynomials with weight α and investigate some interesting identities of these polynomials. Also, we define higher-order degenerate q-Changhee polynomials with weight α and investigate some interesting identities of these polynomials.

2. The degenerate q-Changhee polynomials with weight α

In this section, we introduce degenerate q-Changhee polynomials with weight α which are derived from the fermionic p-adic integral on $\mathbb{Z}_p$ and investigate some properties of these polynomials.

We start with the following definition as.

For $\lambda, t, q \in \mathbb{C}_p$ with $\mid \lambda t \mid < p^{-\frac{1}{p-1}}$ and $\mid 1-q \mid_p < p^{-\frac{1}{p-1}}$. Now, we define degenerate q-Changhee polynomials $Ch_{n,q;\alpha,\lambda}(x)$ are given by

$$\int_{\mathbb{Z}_p} (1 + \lambda \log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y) = \sum_{n=0}^{\infty} Ch_{n,q;\alpha,\lambda}(x)\frac{t^n}{n!} \quad (\alpha > 0). \tag{2.1}$$

When $x = 0$, $Ch_{n,q;\alpha,\lambda} = Ch_{n,q;\alpha,\lambda}(0)$ are called the degenerate q-Changhee numbers with weight α.

Note that

$$\begin{aligned}\lim_{\lambda\to 0} \int_{\mathbb{Z}_p} (1 + \lambda \log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y) &= \sum_{n=0}^{\infty} \lim_{\lambda\to 0} Ch_{n,q;\alpha,\lambda}(x)\frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} (1+t)^{[x+y]_{q^\alpha}} d\mu_{-q}(y) = \sum_{n=0}^{\infty} Ch_{n,q;\alpha}(x)\frac{t^n}{n!},\end{aligned} \tag{2.2}$$

are called the q-Changhee numbers.

Theorem 2.1. *Let $x, y \in \mathbb{C}$ and $n \geq 0$. Then*

$$Ch_{n,q;\alpha,\lambda}(x) = \sum_{j=0}^{n} S_1(n,j) \int_{\mathbb{Z}_p} \binom{\frac{[x+y]_{q^\alpha}}{\lambda}}{j} j! d\mu_{-q}(y)\lambda^j$$

$$= \sum_{j=0}^{m} \int_{\mathbb{Z}_p} \left([x+y]_{q^\alpha}\right)_{j,\lambda} d\mu_{-q}(y), \tag{2.3}$$

where $\left([x+y]_{q^\alpha}\right)_{j,\lambda} = [x+y]_{q^\alpha}([x+y]_{q^\alpha}-\lambda)([x+y]_{q^\alpha}-2\lambda)\cdots([x+y]_{q^\alpha}-(j-1)\lambda)$.

Proof. Using (2.1), we see that

$$\sum_{n=0}^{\infty} Ch_{n,q;\alpha,\lambda}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y)$$

$$= \int_{\mathbb{Z}_p} \sum_{j=0}^{\infty} \binom{\frac{[x+y]_{q^\alpha}}{\lambda}}{j} \lambda^j (\log(1+z))^j d\mu_{-q}(y)$$

$$= \sum_{j=0}^{\infty} \int_{\mathbb{Z}_p} \left([x+y]_{q^\alpha}\right)_{j,\lambda} \sum_{m=j}^{\infty} S_1(m,j) d\mu_{-q}(y)\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{m} \int_{\mathbb{Z}_p} \left([x+y]_{q^\alpha}\right)_{j,\lambda} d\mu_{-q}(y) \right) \frac{t^n}{n!}. \tag{2.4}$$

Comparing the coefficients of on both sides of (2.1) and (2.4), we obtain the result (2.3).

Here, we consider degenerate q-Euler polynomials with weight α which are defined by

$$\sum_{n=0}^{\infty} E_{n,q;\alpha,\lambda}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y). \tag{2.5}$$

Theorem 2.2. *Let* $x, y \in \mathbb{C}$ *and* $n \geq 0$. *Then*

$$E_{n,q;\alpha,\lambda}(x) = \sum_{m=0}^{n} Ch_{m,q;\alpha,\lambda}(x) S_2(n,m). \tag{2.6}$$

Proof. Replacing t by $e^t - 1$ in (2.1), we have

$$\sum_{m=0}^{\infty} Ch_{m,q;\alpha,\lambda}(x)\frac{(e^t-1)^m}{m!} = \int_{\mathbb{Z}_p} (1+\lambda z)^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y)$$

$$= \sum_{n=0}^{\infty} E_{n,q;\alpha,\lambda}(x)\frac{t^n}{n!}. \tag{2.7}$$

On the other hand,

$$\sum_{m=0}^{\infty} Ch_{m,q;\alpha,\lambda}(x)\frac{(e^t-1)^m}{m!} = \sum_{m=0}^{\infty} Ch_{m,q;\alpha,\lambda}(x)\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} Ch_{m,q;\alpha,\lambda}(x)S_2(n,m)\right)\frac{t^n}{n!}. \tag{2.8}$$

In view of (2.7) and (2.8), we get (2.6).

Theorem 2.3. *Let $x, y \in \mathbb{C}$ and $n \geq 0$. Then*

$$Ch_{n,q;\alpha,\lambda}(x) = \sum_{m=0}^{n} E_{m,q;\alpha,\lambda}(x)S_1(n,m). \tag{2.9}$$

Proof. By replacing t by $\log(1+z)$ in (2.5), we get

$$\sum_{m=0}^{\infty} E_{m,q;\alpha,\lambda}(x)\frac{(\log(1+t))^m}{m!}$$

$$= \int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}}d\mu_{-q}(y) = \sum_{n=0}^{\infty} Ch_{n,q;\alpha,\lambda}(x)\frac{t^n}{n!}. \tag{2.10}$$

On the other hand,

$$\sum_{m=0}^{\infty} E_{m,q;\alpha,\lambda}(x)\frac{(\log(1+t))^m}{m!} = \sum_{m=0}^{\infty} E_{m,q;\alpha,\lambda}(x)\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} E_{m,q;\alpha,\lambda}(x)S_1(n,m)\right)\frac{t^n}{n!}. \tag{2.11}$$

By (2.10) and (2.11), we get the result.

Theorem 2.4. *Let $x, y \in \mathbb{C}$ and $n \geq 0$. Then*

$$Ch_{n,q;\alpha,\lambda}(x) = \sum_{j=0}^{n}\sum_{l=0}^{j} S_1(j,l)S_1(n,l)E_{l,q}^{(\alpha)}(x). \tag{2.12}$$

Proof. From (2.1), we note that

$$\sum_{n=0}^{\infty} Ch_{n,q;\alpha,\lambda}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}}d\mu_{-q}(y)$$

$$= \int_{\mathbb{Z}_p} e^{\frac{[x+y]_{q^\alpha}}{\lambda}\log(1+\lambda\log(1+z))} d\mu_{-q}(y)$$

$$= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \left(\frac{[x+y]_{q^\alpha}}{\lambda}\right)^n \sum_{m=n}^{\infty} \lambda^m \frac{(\log(1+z))^m}{m!} d\mu_{-q}(y)$$

$$= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n}\sum_{l=0}^{j} [x+y]_{q^\alpha}^{l} \lambda^{j-l} S_1(j,l) S_1(n,l)\right) d\mu_{-q}(y) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n}\sum_{l=0}^{j} S_1(j,l) S_1(n,l) E_{l,q}^{(\alpha)}(x)\right) \frac{t^n}{n!}. \tag{2.13}$$

Thus, by (2.13), we obtain the result.

Theorem 2.5. *Let $x, y \in \mathbb{C}$ and $n \geq 0$. Then*

$$Ch_{n,q;\alpha,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k} Ch_{m,q;\alpha,\lambda}(x) S_{2,\lambda}(k,m) S_1(n,k). \tag{2.14}$$

Proof. Replacing t by $(1+\lambda\log(1+t))^{\frac{1}{\lambda}} - 1$ in (2.2), we observe that

$$\int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y) = \sum_{m=0}^{\infty} Ch_{m,q;\alpha,\lambda}(x) \frac{1}{m!} [(1+\lambda\log(1+t))^{\frac{1}{\lambda}} - 1]^m$$

$$= \sum_{m=0}^{\infty} Ch_{m,q;\alpha,\lambda}(x) \sum_{k=m}^{\infty} S_{2,\lambda}(k,m) \frac{(\log(1+t))^k}{k!}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{k} Ch_{m,q;\alpha,\lambda}(x) S_{2,\lambda}(k,m) \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n}\sum_{m=0}^{k} Ch_{m,q;\alpha,\lambda}(x) S_{2,\lambda}(k,m) S_1(n,k)\right) \frac{t^n}{n!}. \tag{2.15}$$

In view of (2.1) and (2.15), we complete the proof.

Theorem 2.6. *For $n \geq 0$, $d \in \mathbb{N}$ with $d \equiv 1 (mod\ 2)$ we have*

$$Ch_{n,q;\alpha,\lambda}(x) = \sum_{m=0}^{n}\sum_{k=0}^{m} S_{2,\lambda}(m,k) \sum_{l=0}^{k} S_1(k,l) \lambda^{-l} E_{l,q}^{(\alpha)}(x). \tag{2.16}$$

Proof. From (2.1), we see that

$$\sum_{n=0}^{\infty} Ch_{n,q;\alpha,\lambda}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y)$$
$$= \int_{\mathbb{Z}_p} ((1+\lambda\log(1+t))^{\frac{1}{\lambda}}+1-1)^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y). \tag{2.17}$$

Now, we have

$$((1+\lambda\log(1+t))^{\frac{1}{\lambda}}+1-1)^{\frac{[x+y]_{q^\alpha}}{\lambda}} = \sum_{k=0}^{\infty}\binom{\frac{[x+y]_{q^\alpha}}{\lambda}}{k}((1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1)^k$$
$$= \sum_{k=0}^{\infty}\binom{\frac{[x+y]_{q^\alpha}}{\lambda}}{k}\sum_{m=k}^{\infty} S_{2,\lambda}(m,k)\frac{1}{m!}(\log(1+t))^m$$
$$= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\left(\frac{[x+y]_{q^\alpha}}{\lambda}\right)_k S_{2,\lambda}(m,k)\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m} S_{2,\lambda}(m,k)\sum_{l=0}^{k} S_1(k,l)\lambda^{-l}[x+y]_{q^\alpha}^l\right)\frac{t^n}{n!}.$$

From (2.17), we get

$$\sum_{n=0}^{\infty} Ch_{n,q;\alpha,\lambda}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m} S_{2,\lambda}(m,k)\sum_{l=0}^{k} S_1(k,l)\lambda^{-l}[x+y]_{q^\alpha}^l\right)\frac{t^n}{n!}d\mu_{-q}(y)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{k=0}^{m} S_{2,\lambda}(m,k)\sum_{l=0}^{k} S_1(k,l)\lambda^{-l}\int_{\mathbb{Z}_p}[x+y]_{q^\alpha}^l d\mu_{-q}(y)\right)\frac{t^n}{n!}. \tag{2.18}$$

Thus, by (2.18), we complete the proof.

Theorem 2.7. *Let $x, y \in \mathbb{C}$ and $n \geq 0$. Then*

$$Ch_{n,q;\alpha,\lambda}(x) = \sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{n}{j}\lambda^{k-l}([x]_{q^\alpha})_{m,\lambda}q^{\alpha l x}S_1(k,l)S_1(j,k)S_1(n,m)E_{l,q}^{\alpha}. \tag{2.19}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} Ch_{n,q;\alpha,\lambda}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}}d\mu_{-q}(y). \tag{2.20}$$

By using the result (see [1]), we have

$$[x+y]_{q^\alpha} = \frac{1-q^{\alpha(x+y)}}{1-q^\alpha} = \frac{1-q^\alpha x}{1-q^\alpha} + \frac{q^\alpha x(1-q^\alpha y)}{1-q^\alpha}$$
$$= [x]_{q^\alpha} + q^{\alpha x}[y]_{q^\alpha}. \tag{2.21}$$

Now (2.20), we see that

$$\int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{[x]_{q^\alpha}+q^\alpha[y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y)$$
$$= \int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{[x]_{q^\alpha}}{\lambda}} (1+\lambda\log(1+t))^{\frac{q^\alpha[y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y)$$
$$= \int_{\mathbb{Z}_p} \left(\sum_{j=0}^{\infty} \binom{[x]_{q^\alpha}}{j} \lambda^j (\log(1+t))^j\right)$$
$$\times \left(\sum_{m=0}^{\infty} \frac{q^{m\alpha x}[y]_{q^\alpha}^m (\log(1+\lambda log(1+t)))^m}{\lambda^m m!}\right) d\mu_{-q}(y)$$
$$= \int_{\mathbb{Z}_p} \left(\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} ([x]_{q^\alpha})_{j,\lambda} S_1(n,j)\right) \frac{t^n}{n!}\right)$$
$$\times \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n}\sum_{l=0}^{k} \lambda^{k-l} q^{\alpha l x}[y]_{q^\alpha}^l S_1(k,l) S_1(n,k)\right) \frac{t^n}{n!}\right)$$
$$= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{k=0}^{j}\sum_{l=0}^{k} \binom{n}{j} \lambda^{k-l} ([x]_{q^\alpha})_{m,\lambda} q^{\alpha l x}[y]_{q^\alpha}^l S_1(k,l) S_1(j,k) S_1(n,m)\right) d\mu_{-q}(y) \frac{t^n}{n!}. \tag{2.22}$$

By comparing the coefficients of t, we get (2.19).

Theorem 2.8. *For $n \geq 0$, $d \in \mathbb{N}$ with $d \equiv 1 (mod\ 2)$, we have*

$$Ch_{n,q;\alpha,\lambda}(x) = \sum_{a=0}^{d-1} \frac{1}{[d]_{-q}} \sum_{k=0}^{n}\sum_{j=0}^{k} (-q)^a S_1(k,j) S_1(n,k) [d]_{q^\alpha}^j E_{j,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right) \lambda^{n-k}. \tag{2.23}$$

Proof. Now, we observe that

$$\sum_{n=0}^{\infty} Ch_{n,q;\alpha,\lambda}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y)$$

$$= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{y=0}^{p^N-1} (1+\lambda\log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}} (-q)^y$$

$$= \lim_{N\to\infty} \frac{1}{[dp^N]_{-q}} \sum_{y=0}^{p^N-1} (1+\lambda\log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}} (-q)^{dy}$$

$$= \lim_{N\to\infty} \frac{1}{[d]_{-q}[p^N]_{-q^d}} \sum_{y=0}^{p^N-1} (1+\lambda\log(1+t))^{\frac{[x+a+dy]_{q^\alpha}}{\lambda}} (-q)^{a+dy}$$

$$= \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} \lim_{N\to\infty} \frac{1}{[p^N]_{-q^d}} \sum_{y=0}^{p^N-1} (1+\lambda\log(1+t))^{\frac{[d]_{q^\alpha}[\frac{x+a}{d}+y]_{q^{d\alpha}}}{\lambda}} (-q)^{a+dy}$$

$$= \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} \lim_{N\to\infty} \frac{1}{[p^N]_{-q^d}} \sum_{y=0}^{p^N-1} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n}\sum_{j=0}^{k} S_1(k,j)S_1(n,k)[d]_{q^\alpha}^j \left[\frac{x+a}{d}+y\right]_{q^\alpha}^j \lambda^{n-k} \right) (-q)^{a+dy} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{a=0}^{d-1} \frac{1}{[d]_{-q}} \sum_{k=0}^{n}\sum_{j=0}^{k} (-q)^a S_1(k,j)S_1(n,k)[d]_{q^\alpha}^j E_{j,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right) \lambda^{n-k} \right) \frac{t^n}{n!}. \tag{2.24}$$

Therefore, by (2.24), we obtain the result.

Theorem 2.9. *Let $x, y \in \mathbb{C}$ and $n \geq 0$. Then*

$$Ch_{n,q;\alpha,\lambda}(x+1) + Ch_{n,q;\alpha,\lambda}(x) = [2]_{q^\alpha} \sum_{m=0}^{n} ([x]_{q^\alpha})_{m,\lambda} \lambda^m S_1(n,m). \tag{2.25}$$

Proof. From (1.1), we have

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = [2]_q f(0). \tag{2.26}$$

Therefore by (2.1) and (2.26), we have

$$q\int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{[x+1+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y) + \int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_{-q}(y)$$

$$= [2]_{q^\alpha} (1+\lambda\log(1+t))^{\frac{[x]_{q^\alpha}}{\lambda}}. \tag{2.27}$$

By (2.1) and (2.27), we get

$$\sum_{n=0}^{\infty}\left(Ch_{n,q;\alpha,\lambda}(x+1)+Ch_{n,q;\alpha,\lambda}(x)\right)\frac{t^n}{n!}$$

$$=[2]_{q^\alpha}\sum_{m=0}^{\infty}([x]_{q^\alpha})_{m,\lambda}\lambda^m\frac{1}{m!}(\log(1+t))^m$$

$$=[2]_{q^\alpha}\sum_{m=0}^{\infty}([x]_{q^\alpha})_{m,\lambda}\lambda^m\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}$$

$$=[2]_{q^\alpha}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}([x]_{q^\alpha})_{m,\lambda}\lambda^m S_1(n,m)\right)\frac{t^n}{n!}. \tag{2.28}$$

Comparing the coefficients of t on both sides, we get (2.25).

For $r\in\mathbb{N}$, we define the higher-order degenerate q-Changhee polynomials of the second kind with weight α which are given multivariate fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{[x+x_1+\cdots+x_r]_{q^\alpha}}{\lambda}}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$

$$=\sum_{n=0}^{\infty}Ch^{(r)}_{n,q;\alpha;\lambda}(x)\frac{t^n}{n!},\ (n\geq 0). \tag{2.29}$$

When $x=0, Ch^{(r)}_{n,q;\alpha;\lambda}=Ch^{(r)}_{n,q;\alpha;\lambda}(0)$ are called the higher-order degenerate q-Changhee numbers of the second kind with weight α.

Theorem 2.10. *For $x,y\in\mathbb{C}$, $n\geq 0$ and $r\in\mathbb{N}$, we have*

$$Ch^{(r)}_{n,q;\alpha;\lambda}(x)=\sum_{m=0}^{n}\sum_{k=0}^{m}S_1(m,k)S_1(n,m)\lambda^{n-m}E^{(r)}_{k,q;\alpha}.$$

Proof. From (2.29), we note that

$$\sum_{n=0}^{\infty}Ch^{(r)}_{n,q;\alpha;\lambda}(x)\frac{t^n}{n!}=$$

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log_\lambda(1+t))^{\frac{[x+x_1+\cdots+x_r]_{q^\alpha}}{\lambda}}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)$$

$$= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\frac{[x_1+\cdots+x_r+x]_{q^\alpha}}{\lambda}}{m} \lambda^m (\log(1+t)))^m d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)$$

$$= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} ([x_1+\cdots+x_r+x]_{q^\alpha})_{\lambda,m} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \frac{1}{m!} (\log(1+t)))^m$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} ([x_1+\cdots+x_r+x]_{q^\alpha})_{\lambda,m} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) S_1(n,m) \right) \frac{t^n}{n!}$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{m=0}^{\infty} ([x_1+\cdots+x_r+x]_{q^\alpha})_{\lambda,m} \sum_{n=m}^{\infty} S_1(n,m) d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \frac{t^n}{n!}$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{k=0}^{m} [x_1+\cdots+x_r+x]_{q^\alpha}^k \lambda^{n-m} S_1(n,m) \right) d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{k=0}^{m} S_1(m,k) S_1(n,m) \lambda^{n-m} E_{k,q;\alpha}^{(r)} \right) \frac{t^n}{n!}. \tag{2.30}$$

Therefore, by (2.30), we obtain the result.

Theorem 2.11. *Let $x, y \in \mathbb{C}$ and $n \geq 0$. Then*

$$E_{n,\lambda}^{(r)}(x) = \sum_{m=0}^{n} \widehat{Ch}_{m,\lambda}^{(r)}(x) S_{2,\lambda}(n,m).$$

Proof. By changing t by $e^t - 1$ in (2.29), we have

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+x_1+\cdots+x_r]_{q^\alpha}}{\lambda}} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r)$$

$$= \sum_{n=0}^{\infty} E_{n,q;\alpha,\lambda}^{(r)} \frac{t^n}{n!}. \tag{2.31}$$

On the other hand, we have

$$= \sum_{m=0}^{\infty} Ch_{m,q;\alpha,\lambda}^{(r)}(x) \frac{(e^t-1)^m}{m!}$$

$$= \sum_{m=0}^{\infty} Ch_{m,q;\alpha,\lambda}^{(r)}(x) \sum_{n=m}^{\infty} S_2 n, m \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} Ch_{m,q;\alpha,\lambda}^{(r)}(x) S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}. \tag{2.32}$$

Therefore, by (2.31) and (2.32), we get the result.

3. Conclusion

In this article, we defined degenerate q-Changhee polynomials and numbers with weight α which were actually called the degenerate q-Changhee polynomials and numbers introduced by Kim *el al.* [14]. We derived their explicit expressions and some identities involving them. Further, we introduced the higher-order degenerate q-Changhee polynomials and numbers and deduced their explicit expressions and some identities related to them.

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